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DISTRIBUTION MODULO 1 OF SOME OSCILLATING SEQUENCES, II

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ABSTRACT

Sequences defined by means of polynomials and highly differentiable quasi-periodic functions are considered. It is proved that under some conditions such sequences must assume small values modulo 1, or even be dense modulo 1. Negative results, demonstrating that some differentiablity conditions are necessary, are also obtained.

1. Introduction

The theory of diophantine approximations is quite well developed for polynomials. For example, one knows that, given a polynomial P with real coefficients, the values P assumes at the positive integer points come, modulo 1, arbitrarily

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close to $P(0)$. More specifically, there exists a constant $\theta > 0$, depending only on $\deg P$, such that the inequality

$$\|P(n) - P(0)\| < \frac{1}{n^\theta}$$

(where $\|t\|$ denotes the distance of $t \in \mathbb{R}$ from the nearest integer) has infinitely many solutions $n \in \mathbb{N}$ (see, for example, [3, Th. 4.5, Th. 5.2]). Analogous results hold for the case of several polynomials [1, Th. 1], [2, (6), Th. 3]. We mention, however, that the best possible value of θ in (1.1) is still unknown for non-linear polynomials. It is also well known that, if P has at least one irrational coefficient (not counting the free term), then the sequence $(P(n))_{n=1}^\infty$ is uniformly distributed modulo 1 (henceforward - u.d.).

Some results along these lines were obtained also for more general functions; see, for example, [12]. More generally, in [7] necessary and sufficient conditions for uniform distribution and for density modulo 1 were given for large families of sequences defined by means of "natural" formulas. All these results apply, however, to functions satisfying some monotonicity conditions, while the case of oscillating sequences is more difficult to deal with. A metrical result, due to LeVeque [13], asserts that for any increasing sequence a_n of integers the sequence $a_n \cos a_n\alpha$ is u.d. for almost every α. Of course, this is not the case in general (i.e., for an arbitrary sequence a_n) for every α. One may inquire whether for some special sequences a_n the metrical result can be replaced by a global result. Furstenberg and Weiss [10] proved that for almost every α the set of solutions n of the inequality $\|n \cos n\alpha\| < \varepsilon$ is not of bounded gaps for every $\varepsilon < \frac{1}{2}$ (unlike the case of polynomials, where the set of solutions n of the inequality $\|P(n) - P(0)\| < \varepsilon$ is of bounded gaps for every $\varepsilon > 0$). They raised the question whether this inequality has a solution for every α and $\varepsilon > 0$.

This question was settled in [4], with further refinements obtained in [8]. It turns out that, not only does the inequality necessarily have solutions, but actually the sequence $(n \cos n\alpha)$ is necessarily u.d. unless α is a rational multiple of π. Moreover, the results of these papers apply to sequences of the form $P(n)f(Q(n))$, where P and Q are arbitrary polynomials and f a periodic "highly differentiable" function. Specifically, there exist (effective) positive numbers s and ρ, depending only on the degrees of P and Q, such that, if f is s times differentiable at the point $Q(0)$, then the inequality

$$\|P(n)f(Q(n)) - P(0)f(Q(0))\| < \frac{1}{n^\rho} \tag{1.1}$$

has infinitely many positive integer solutions n [8, Th. 2.2]. Thus, for example, the inequalities

$$\|n\beta\cos n\alpha\| < \frac{C}{n^{\frac{1}{5}}} \tag{1.2}$$

(C being a certain absolute constant) and

$$\|n\beta\sin n\alpha\| < \frac{1}{n^{\frac{2}{13}-\varepsilon}} \tag{1.3}$$

($\varepsilon > 0$ arbitrary) have infinitely many solutions. In some cases it is possible to show that the sequence $P(n)f(Q(n))$ is dense modulo 1, or even u.d. [8, Th. 3.1, 3.2, 4.1, 4.2].

In this paper we improve some of the results of [8] in two respects. First, we want to consider the sequence $P(n)f(Q(n))$ with functions f which are not necessarily periodic. It turns out to be possible to deal with quasi-periodic functions (see Section 2). Another direction of extension, motivated by [10], is looking at sequences of the form $P(n)f(Q(n)g(R(n)))$, where P, Q and R are polynomials and f and g periodic. We obtain positive results in both of these directions. We mention in particular that, for the second case we can sometimes prove density modulo 1 (but not uniform distribution) by means of exponential sums. The density result is thus quantitative, namely we obtain a lower bound on the number of terms of the sequence belonging, modulo 1, to various intervals.

We also present an ad-hoc simple method which enables us to improve upon (1.2) (but not (1.3)).

On the other hand, one may ask to what extent the smoothness conditions imposed on f, already in the case of the sequence $P(n)f(Q(n))$, are in fact necessary. Obviously, some continuity condition is inevitable. We prove a strong negative result, which implies as a very special case that f may be differentiable any finite number of times, yet the small values result will fail for appropriately chosen P and Q.

The main results are stated in Section 2. Section 3 deals with the proofs of the small values results, Section 4 – with the proofs of the density results, and Section 5 – with those of the negative results.

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2. The main results

Recall that a function $f\colon \mathbb{R} \to \mathbb{R}$ is **quasi-periodic** if it is the diagonal of a periodic function of several variables, i.e.,

$$f(x) = F(x, x, \ldots, x),$$

where $F\colon \mathbb{R}^r \longrightarrow \mathbb{R}$ is periodic in each variable. Alternatively, we can write f in this case in the form

$$f(x) = F(\alpha_1 x, \alpha_2 x, \ldots, \alpha_r x),$$

where F is of period 1 in each variable. The family of quasi-periodic functions extends that of periodic functions, and it is thus natural to consider sequences of the form $P(n)f(Q(n))$, where f is quasi-periodic. In the following theorem we take one step further.

THEOREM 2.1: *Let $P, Q_1, Q_2, \ldots, Q_r$ be polynomials of degrees $d, e_1, e_2, \ldots, e_r$, respectively. Write:*

$$Q_j(x) = \sum_{i=0}^{e_j} c_{ij} x^i, \qquad 1 \le j \le r.$$

Suppose that

$$(2.1) \qquad \|c_{ij} n_k^i\| < \frac{1}{n_k^{\tau_{ij}}}, \qquad 1 \le j \le r, \quad 1 \le i \le e_j, \quad k = 1, 2, \ldots,$$

for some sequence (n_k) and constants $\tau_{ij} > 0$, $1 \le j \le r, 1 \le i \le e_j$. Let $F\colon \mathbb{R}^r \longrightarrow \mathbb{R}$ be periodic in each variable. Assume that F is differentiable of order L at the point $(Q_1(0), Q_2(0), \ldots, Q_r(0))$, where $L > \frac{d}{\min_{i,j} \tau_{ij}} - 1$. Put $E = \max_{1 \le j \le r} e_j$. Let $\theta > 0$ be such that, if T is any polynomial with $\deg T = d + (L-1)E$, then for every sufficiently large N the inequality $\|T(n) - T(0)\| < \frac{1}{N^\theta}$ has a solution $1 \le n \le N$. Denote:

$$\gamma_0 = \min_{1 \le j \le r, 1 \le i \le e_j} \frac{\tau_{ij} L - d}{d + \beta + iL}.$$

Then the inequality

$$(2.2) \quad \|P(n)F(Q_1(n), \ldots, Q_r(n)) - P(0)F(Q_1(0), \ldots, Q_r(0))\| < \frac{K}{n^{\theta\gamma_0/(1+\gamma_0)}},$$

with a suitable constant K, has infinitely many solutions n.

In view of [1, Th. 1], [2, (6), Th. 3] this implies

THEOREM 2.2: *Given positive integers D and r, there exist (effective) numbers $L \in \mathbb{N}$ and $\rho > 0$ such that the following holds. If $P, Q_1, Q_2, \ldots, Q_r$ are any polynomials of degrees bounded by D and $F\colon \mathbb{R}^r \longrightarrow \mathbb{R}$ a function periodic in each variable and differentiable of order L at a neighbourhood of the point $(Q_1(0), Q_2(0), \ldots, Q_r(0))$, then (2.2) has infinitely many solutions n.*

Clearly, Theorems 2.1 and 2.2 yield corresponding results for functions of the form $P(x)f(Q(x))$, where f is quasi-periodic. Another consequence is

COROLLARY 2.1: *Let $P, Q_1, Q_2, \ldots, Q_r$ be polynomials and $f_1, f_2, \ldots, f_r$ periodic functions. If each f_j is sufficiently many times (in terms of the degrees of the polynomials) differentiable, then there exists a $\rho > 0$ such that the inequality*

$$\|P(n)f_1(Q_1(n))\ldots f_r(Q_r(n)) - P(0)f_1(Q_1(0))\ldots f_r(Q_r(0))\| < \frac{1}{n^\rho}$$

has infinitely many solutions n.

The results are easiest to apply when all considered functions are infinitely differentiable.

Example 2.1: For every positive integer r there exists a $\rho = \rho(r) > 0$ such that, for any $\alpha_1, \alpha_2, \ldots, \alpha_r \in \mathbb{R}$, the inequality

$$\|n \cos n\alpha_1 \cdots \cos n\alpha_r\| < \frac{1}{n^\rho}$$

has infinitely many solutions n.

The following proposition deals with a very specific case, but enables to improve the result of [8] on small values of the sequence $n\beta \cos n\alpha$.

PROPOSITION 2.1: *Suppose f is periodic, twice differentiable at 0 and $f'(0) = 0$. Then there exists a constant C such that the inequality*

$$\|nf(n\alpha)\| < \frac{C}{n^{1/3}}$$

has infinitely many solutions n.

Example 2.2: For every $\alpha, \beta \in \mathbb{R}$ there exists a constant C such that the inequality

$$\|n\beta \cos n\alpha\| < \frac{C}{n^{1/3}} \tag{2.3}$$

has infinitely many solutions n.

Remark 2.1: While the exponent $\frac{1}{3}$ in the denominator of the right hand side of (2.3) improves upon the $\frac{1}{5}$ obtained in [8, Prop. 2.1], it is probably still far from the best possible, which may be $1-\varepsilon$ or even 1.

Note that Proposition 2.1 does not apply to prove an analogue of (2.3), with the function cos replaced by sin. In Remark 3.1 we shall explain why the method used to prove the proposition fails in this case.

THEOREM 2.3: *Let P be a polynomial of degree $d \geq 1$, f_1 and f_2 non-constant functions with period 1. Assume that $f_2'(x_0) = f_2''(x_0) = \ldots = f_2^{(l-1)}(x_0) = 0$, $f_2^{(l)}(x_0) \neq 0$, for some $x_0 \in [0,1]$ and $l \geq 2$, that the functions f_1 and f_2 are differentiable at least $\frac{1}{2} + \frac{(7l+1)d}{l-1}$ times in some neighborhoods of the points 0 and x_0, respectively, and that $f_1^{(s)}(0) \neq 0$ for some $s \geq \frac{d}{l-1} + \frac{1}{2(l+1)}$. Then for every irrational α the sequence $(P(n)f_1(nf_2(n\alpha)))_{n=1}^{\infty}$ is dense modulo 1.*

THEOREM 2.4: *Let P be a polynomial of degree $d \geq 1$ and f_1 and f_2 non-constant periodic functions with period 1. Assume there exists a point $x_0 \in [0,1)$ such that $f_2'(x_0) = f_2''(x_0) = \ldots = f_2^{(l-1)}(x_0) = 0 \neq f_2^{(l)}(x_0)$, where $l \geq 2$. Let $\epsilon_0 \leq \min\{\frac{1}{2l-1}, \frac{1}{4}\}$ be a positive number, with the functions f_1 and f_2 being $[2d(1+\frac{1}{\epsilon_0}) + \frac{10}{3}]$ times differentiable in the whole interval $[0,1)$ and in a neighborhood of x_0, respectively. Assume also that*

$$|f_1^{(j)}(x)| + |f_1^{(j+1)}(x)| + |f_1^{(j+2)}(x)| \geq a_0, \qquad j \geq 9d, \quad x \in [0,1),$$

for some $a_0 > 0$. Let I be any non-trivial subinterval of $[0,1)$, and $\alpha = \frac{p}{q} + \eta$ with $|\eta| \leq 1/q^2$. Then:

$$|\{n\colon 1 \leq n \leq q^{1+\epsilon_0}, \langle P(n)f_1(nf_2(n\alpha))\rangle \in I\}| \gg q^{1-\frac{29\epsilon_0}{19}}.$$

COROLLARY 2.2: *In the setup of Theorem 2.4, the sequence $P(n)f_1(nf_2(n\alpha))$ is dense modulo 1 for every irrational α.*

COROLLARY 2.3: *Let $f(x) = P(x)f_1(xf_2(x\alpha))$ be as in Theorem 2.4. Suppose α is a real number for which there exists a constant $u > 0$ such that*

$$\max\{q\colon q \leq N,\ \|q\alpha\| \leq 1/N\} \gg N^u.$$

Then for any non-trivial interval $I \subseteq [0,1)$

$$|\ \{n \leq N\colon \langle f(n)\rangle \in I\}\ | \gg N^{u-3\epsilon_0}.$$

PROPOSITION 2.3: *Consider the sequence* $n\cos(n\cos n\alpha)$.

1. *If* $\frac{\alpha}{\pi}$ *is irrational, then the sequence is dense modulo* 1; *moreover, for any non-trivial interval* $I \subseteq [0,1)$,
$$|\{n\colon 1 \le n \le N; \langle n\cos(n\cos n\alpha)\rangle \in I\}| \gg N^{2/3}.$$
2. *If* $\frac{\alpha}{\pi}$ *is rational, write* $\alpha = \frac{p}{q}\pi$, *where* $(p,q) = 1$. *Then:*
   i. *If* q *is odd then the sequence is u.d.*
   ii. *If* q *is even then the sequence is distributed according to a convex combination of the Lebesgue measure on* $[0,1)$ *and the Dirac measure at 0.*

In particular, if $\frac{\alpha}{\pi}$ *is rational then the sequence is dense modulo 1 and for every non-trivial interval* $I \subseteq [0,1)$
$$|\{n\colon 1 \le n \le N; \langle n\cos(n\cos n\alpha)\rangle \in I\}| \gg N\,.$$

It is probably possible to replace the polynomials, at least in some of our results, by more general "nice" functions having regular growth at infinity, e.g. belonging to a Hardy field (see [5], [6], [7]). In this paper we shall give only a single example of this kind.

PROPOSITION 2.4: *For every* $\alpha \in \mathbb{R}$ *the sequence* $\log n \cos n\alpha$ *is dense modulo 1.*

Remark 2.2: It can be shown that there exist uncountably many numbers α for which this sequence is not u.d. The same is true if $\log n$ is replaced by any "nice" function (in the sense mentioned prior to Proposition 2.4) approaching infinity slower (say, $\sqrt{\log n}$ or $\log\log n$). However, our proof of this fact fails if $\log n$ is replaced by $(\log n)^{1+\varepsilon}$.

THEOREM 2.5: *Let* $(\alpha_n)_{n\ge1}$, $(\beta_n)_{n\ge1}$, $(\mu_n)_{n\ge1}$ *be three sequences of real numbers. Assume that*

1. $\mu_n \ne 0$ *for all* n;
2. $\alpha_m \ne \alpha_n \pmod 1$ *for* $m \ne n$;
3. $\lim_{n\to\infty} |\mu_n| = \infty$.

Then there exists a continuous 1*-periodic function* $f(x)$ *such that*
$$\mu_n f(\alpha_n) = \beta_n \pmod 1 \tag{2.4}$$
for all $n \ge 1$.

The following shows that some differentiability conditions on f are indispensable in Theorem 2.1 (even if the function is required to be periodic).

COROLLARY 2.4: *Let P and Q be non-constant polynomials, with Q having at least one irrational coefficient (besides the constant term). Then there exists a continuous 1-periodic function f such that*

$$\|P(n)f(Q(n)) - P(0)f(Q(0))\| = \frac{1}{2}$$

for all sufficiently large n.

The corollary follows immediately upon observing that the conditions guarantee that all the values $(Q(n))$ are distinct modulo 1 from some place on.

THEOREM 2.6: *Let $(\alpha_n)_{n\geq 1}$, $(\beta_n)_{n\geq 1}$, $(\mu_n)_{n\geq 1}$ be three sequences of real numbers, let $t \geq 1$, $d > 0$ and p be real numbers, and let k be a positive integer. Suppose that*

1. *$\mu_n \neq 0$ for all n;*
2. *$\|\alpha_m - \alpha_n\| > dm^{-t}$ for all integers $m > n \geq 1$;*
3. *$\liminf_{n\longrightarrow\infty} \left|\frac{\mu_n}{n^p}\right| > 0$;*
4. *$p > kt$.*

   Then there exists a 1-periodic C^k function $f(x)$ on $\mathbb{R}$ such that (2.4) holds for all $n \geq 1$.

The proofs of the small values results depend on the same result for polynomials. As these results seem to be currently quite far from the best possible, one would expect a sizeable gap between the differentiability assumptions on f guaranteeing that $\|P(n)f(Q(n)) - P(0)f(Q(0))\|$ assumes small values and the counter-examples we can construct. It makes sense therefore to make the comparison in the case this obstacle is missing, namely for linear Q. To this end, we first make the following

Definition 2.1: An irrational number α is **typical** if for every $\epsilon > 0$ the inequality

$$\|n\alpha\| < \frac{1}{n^{1+\epsilon}}$$

has only a finite number of solutions n.

Note that it is well known that Lebesgue almost all real numbers, as well as all irrational algebraic numbers, are typical.

COROLLARY 2.5: *Let $(\alpha_n)_{n\geq 1}$ be defined by $\alpha_n = \langle n\alpha\rangle$, $n \geq 1$, for some typical number α. Let $k \geq 1$ be an integer. Then for any sequence $(\mu_n)_{n\geq 1}$ with $\mu_n \neq 0$ such that*

$$\liminf_{n\longrightarrow\infty} \frac{\log|\mu_n|}{\log n} > k \tag{2.5}$$

and for any real sequence $(\beta_n)_{n\geq 1}$ there exists a 1-periodic C^k function $f(x)$ on $\mathbb{R}$ such that (2.4) holds for all $n \geq 1$.

Remark 2.3: It can be shown in a similar fashion that if condition (2.5) is replaced by

$$\lim_{n\longrightarrow\infty} \frac{\log|\mu_n|}{\log n} = +\infty$$

then $f(x)$ can be chosen to be C^∞.

Another consequence of the foregoing is

COROLLARY 2.6: *Let $\deg P = d$ and $Q(x) = \alpha x$. Then:*

1. *If f is 1-periodic C^d function, then the inequality*

   $$\|P(n)f(Q(n)) - P(0)f(Q(0))\| < \varepsilon$$

   has infinitely many solutions for every $\varepsilon > 0$.
2. *If α is typical, then there exists a 1-periodic C^{d-1} function f such that*

   $$\|P(n)f(Q(n)) - P(0)f(Q(0))\| = \frac{1}{2}, \qquad n = 1, 2, \dots .$$

In fact, the first part follows from the discussion in [4, Sec. 2], the second - from Corollary 2.5.

Thus, in the sense discussed above, the gap in the differentiability conditions is very small. We raise the following

QUESTION: *Do there exist a 1-periodic continuously differentiable non-constant function f on $\mathbb{R}$ and an irrational α such that $nf(n\alpha)$ is an integer for every n?*

Remark 2.4: By techniques similar to those used in the proof of Theorem 2.5, one can easily show that such an f does exist in the following two cases:

1. The condition on f is relaxed by only requiring that it satisfies Lipschitz condition (α may be any badly approximable number).
2. The sequence is replaced by $n^t f(n\alpha)$ with $t > 1$ (α may be any typical number).

3. Proofs of the small values results

Given $t \in \mathbb{R}$, denote by $\{t\}$ the unique number in $[-\frac{1}{2}, \frac{1}{2})$ with $t - \{t\} \in \mathbb{Z}$.

Proof of Theorem 2.1: We may assume that F is of period 1 in each variable and that all the polynomials Q_j are without constant term. Express F in some neighbourhood of $(0, 0, \ldots, 0)$ in the form

$$F(x_1, x_2, \ldots, x_r) = \sum_{0 \le l_1 + \ldots + l_r < L} a_{l_1, \ldots, l_r} x_1^{l_1} \ldots x_r^{l_r} + O\left(\max |x_j|^L\right)$$

for suitable constants $a_{l_1,\ldots,l_r}$. Let (n_k) be a sequence of solutions of (2.2) and (h_k) an arbitrary sequence of positive integers. Putting

$$f(x) = F(Q_1(x), \ldots, Q_r(x)), \quad g(x) = P(x) f(x),$$

we have

$$\begin{aligned} P(h_k n_k) F(Q_1(h_k n_k), \ldots, Q_r(h_k n_k)) - P(0) F(Q_1(0), \ldots, Q_r(0)) \\ = g(h_k n_k) - g(0), \end{aligned}$$

and it has to be shown that, if (h_k) is chosen appropriately, then $g(h_k n_k) - g(0)$ is very small modulo 1. Set $d = \deg P$. Then:

$$\begin{aligned} f(h_k n_k) &= F\left(\{Q_1(h_k n_k)\}, \ldots, \{Q_r(h_k n_k)\}\right) \\ &= F\left(\sum_{i=1}^{e_1} h_k^i \{c_{i1} n_k^i\}, \ldots, \sum_{i=1}^{e_r} h_k^i \{c_{ir} n_k^i\}\right) \\ &= \sum_{0 \le l_1 + \ldots + l_r < L} a_{l_1, \ldots, l_r} \left(\sum_{i=1}^{e_1} h_k^i \{c_{i1} n_k^i\}\right)^{l_1} \cdots \left(\sum_{i=1}^{e_r} h_k^i \{c_{ir} n_k^i\}\right)^{l_r} \\ &\qquad + O\left(\left(\max_{1 \le j \le r} \sum_{i=1}^{e_j} h_k^i \|c_{ij} n_k^i\|\right)^L\right) \\ &= F(0, \ldots, 0) + S(h_k) + O\left(\left(\max_{1 \le j \le r} \sum_{i=1}^{e_j} h_k^i \|c_{ij} n_k^i\|\right)^L\right), \end{aligned}$$

where S is a polynomial without free term of degree $(L-1)E$ in h_k whose coefficients are themselves polynomials in n_k and $\{c_{ij} n_k^i\}$, $1 \le j \le r, 1 \le i \le e_j$. Therefore

$$g(h_k n_k) - g(0) = S_1(h_k) + P(h_k n_k) O\left(\left(\max_{1 \le j \le r} \sum_{i=1}^{e_j} h_k^i \|c_{ij} n_k^i\|\right)^L\right), \tag{3.1}$$

where S_1 is of degree $d+(L-1)E$ in h_k. Let (H_k) be a sequence of real numbers with $H_k \longrightarrow \infty$, to be determined later. Then for each sufficiently large k the inequality

$$S_1(h_k) < \frac{1}{H_k^\theta} \tag{3.2}$$

has a solution h_k with $1 \le h_k \le H_k$. For the last term on the right hand side of (3.1) we have:

$$\begin{aligned} P(h_k n_k) \cdot O\left(\left(\max_{1\le j\le r} \sum_{i=1}^{e_j} h_k^i \|c_{ij} n_k^i)\|\right)^L\right) &\ll H_k^d n_k^d \max_{\substack{1\le j\le r\\ 1\le i\le e_j}} \left(H_k^{iL} \|c_{ij} n_k^i\|^L\right) \\ &\ll \max_{\substack{1\le j\le r\\ 1\le i\le e_j}} \left(H_k^{d+iL} n_k^{d-\tau_{ij}L}\right). \end{aligned}$$

Taking $H_k = n_k^\gamma$, where $\gamma > 0$ will be specified later, we obtain:

$$P(h_k n_k) \cdot O\left(\left(\max_{1\le j\le r} \sum_{i=1}^{e_j} h_k^i \{c_{ij} n_k^i)\}\right)^L\right) \ll n_k^{\max_{1\le j\le r, 1\le i\le e_j}(d+\gamma d + i\gamma L - \tau_{ij}L)}. \tag{3.3}$$

Substituting (3.2) and (3.3) in (3.1), we arrive at:

$$\|g(h_k n_k) - g(0)\| \ll n_k^{-\theta\gamma} + n_k^{\max_{1\le j\le r, 1\le i\le e_j}(d+\gamma d + i\gamma L - \tau_{ij}L)}. \tag{3.4}$$

Since $\frac{d}{\tau_{ij}} < L$ for any i and j, for sufficiently small $\gamma > 0$ the exponents in both terms on the right hand side of (3.4) are negative. Upon increasing γ, the first term improves, while the second term becomes worse.

The optimal choice for γ is obtained when for the first time

$$\theta\gamma = \tau_{ij}L - i\gamma L - d - \gamma d,$$

namely for

$$\gamma_0 = \min_{1\le j\le r, 1\le i\le e_j} \frac{\tau_{ij}L - d}{d+\theta+iL}.$$

This finally yields

$$\|g(h_k n_k) - g(0)\| < \frac{C}{n_k^{\theta\gamma_0}} < \frac{K}{(h_k n_k)^{\theta\gamma_0/(1+\gamma_0)}},$$

thus proving the theorem. ∎

Proof of Proposition 2.1: Since $f'(0) = 0$, we have for a suitable $a \in \mathbb{R}$

$$f(x) = a + O(x^2)$$

as $x \longrightarrow 0$. It follows from [11, p. 43, Th. 3] that, given any $0 \le \mu \le 1$, the system

$$\|na\| < \frac{1}{n^{\mu}}, \quad \|n\alpha\| < \frac{1}{n^{1-\mu}}$$

has infinitely many solutions n. As $n \longrightarrow \infty$ along this set we have

$$\{nf(n\alpha)\} = \{n\left(a + O\left(\{n\alpha\}^2\right)\right)\} \ll \|na\| + n\|n\alpha\|^2,$$

and therefore

$$\|nf(n\alpha)\| \le \frac{1}{n^{\mu}} + n \cdot C_1 \frac{1}{n^{2-2\mu}} \le \frac{1}{n^{\mu}} + \frac{C_1}{n^{1-2\mu}}$$

for an appropriate constant C_1. With the choice $\mu = \frac{1}{3}$, the proposition follows. ∎

Remark 3.1: Examining the proof of Proposition 2.1 it is clear that the analogue of (2.3), with $n\beta \cos n\alpha$ replaced by $n\beta \sin n\alpha$, could be proved in the same way if it would be true that the system of inequalities

$$\left\|n\frac{\alpha}{\pi} - \frac{1}{2}\right\| < \frac{1}{n^{\frac{2}{3}}}, \quad \|n\beta\| < \frac{1}{n^{\frac{1}{3}}}$$

has infinitely many solutions n (which seems plausible if both $\frac{\alpha}{\pi}$ and $\frac{\beta\pi}{\alpha}$ are irrational). However, we recall that, although in the 1-dimensional case one can obtain the same results for inhomogeneous approximations as for homogeneous, in the case of simultaneous approximations the quantitative analogues are no longer true. Namely, for any x the inequality $\|nx\| < 1/n$ has infinitely many solutions, and so has the inequality $\|nx - x_0\| < 1/n$ for arbitrary fixed x_0 if x is irrational. However, whereas the system

$$\|nx\| < \frac{1}{\sqrt{n}}, \quad \|ny\| < \frac{1}{\sqrt{n}}$$

has infinitely many solutions for any x, y, there exist x, y, x_0, y_0, with $1, x, y$ linearly independent over $\mathbb{Q}$, such that the corresponding inhomogeneous system

$$\|nx - x_0\| < \frac{1}{\sqrt{n}}, \quad \|ny - y_0\| < \frac{1}{\sqrt{n}},$$

has but finitely many solutions (even upon replacing $\frac{1}{\sqrt{n}}$ by a function $\varepsilon(n)$ converging to 0 arbitrarily slowly) [9, Th. V.XV].

4. Proofs of the density results

We shall need a few lemmas. The following was proved in [8, Lemma 4.1].

LEMMA 4.1: *Let $f \in C^j[X_1, X_2]$, and suppose that $0 < \lambda \le f^{(j)}(x) \ll \lambda$. Then*

$$\Big| \sum_{X_1 \le x \le X_2} (f(x)) \Big| \ll X\lambda^{1/(J-2)} + X^{1-2/J} + X(\lambda X^{4-8/J})^{-2/J} + 1,$$

where $X = X_2 - X_1$ and $J = 2^j$, the implied constants depending only on j.

LEMMA 4.2: *Let*

$$N = N(u,v) = \big| \{x \in [X, 2X) \colon 0 \le u \le \{f(x)\} \le u + \Delta = v \le 1\} \big| .$$

Then for any $\delta \le \Delta/4$

$$\big| N - X\Delta \big| \le X\delta + 2\sum_{j=1}^{\infty} \min\Big\{\Delta, \frac{\Delta}{\delta^2 j^2}, \frac{1}{\delta^2 j^3}\Big\} \Big| \sum_{X \le x < 2X} e(jf(x)) \Big| .$$

Proof: Denoting by $\chi_{[a,b]}$ the characteristic function of $[a, b]$ we obtain

$$\begin{aligned}(4.1)\qquad &(2\delta)^{-2} \sum_{X \le x < 2X} \int_{-\delta}^{\delta}\int_{-\delta}^{\delta} \chi_{[u+2\delta, v-2\delta]}(f(x) + t_1 + t_2)\, dt_2 dt_1 \le N \\ &\le (2\delta)^{-2} \sum_{X \le x < 2X} \int_{-\delta}^{\delta}\int_{-\delta}^{\delta} \chi_{[u-2\delta, v+2\delta]}(f(x) + t_1 + t_2)\, dt_2 dt_1.\end{aligned}$$

Expanding $\chi_{[a,b]}$ into a Fourier series, we get

$$(4.2)\qquad (2\delta)^{-2} \int_{-\delta}^{\delta}\int_{-\delta}^{\delta} \chi_{[a,b]}(f(x) + t_1 + t_2) dt_2 dt_1 = b - a + \sum_{|j|=1}^{\infty} a_j e(jf(x)),$$

where

$$a_j = e(\frac{-j(b+a)}{2})\frac{\sin \pi j(b-a)}{\pi j} \cdot \left(\frac{\sin 2\pi j\delta}{2\pi j\delta}\right)^2,$$

and therefore

$$(4.3)\qquad |a_j| \le \min\{b - a, \frac{b-a}{4\pi^2 j^2 \delta^2}, \frac{1}{4\pi^3 \delta^2 j^3}\}.$$

Substituting (4.2) in (4.1) and using (4.3), we easily complete the proof. ∎

LEMMA 4.3: *There exist constants $0 < c_1 < c_2$ such that for every irrational number α the inequality*

$$\frac{c_1}{q^2} \leq \left|\alpha - \frac{p}{q}\right| \leq \frac{c_2}{q^2} \tag{4.4}$$

has infinitely many solutions $\frac{p}{q}$ with $(p,q) = 1$.

Proof: We prove it, say, for $c_1 = \frac{1}{3}$, $c_2 = \frac{10}{3}$. The lemma is well-known if we take $c_1 = 0$, $c_2 = \frac{1}{\sqrt{5}}$. Assume that for some α we have only finitely many p, q with $(p,q) = 1$ satisfying (4.4). Take sufficiently large p_1 and q_1 with

$$\left|\alpha - \frac{p_1}{q_1}\right| \leq \frac{1}{q_1^2}, \quad (p_1, q_1) = 1.$$

From the elementary theory of Farey fractions it follows that we can find $p_1 < p_2 < 2p_1$, $q_1 < q_2 < 2q_1$ for which $p_1q_2 - p_2q_1 = 1$. Then:

$$\begin{aligned}\left|\alpha - \frac{p_2}{q_2}\right| &\geq \left|\frac{p_1}{q_1} - \frac{p_2}{q_2}\right| - \left|\alpha - \frac{p_1}{q_1}\right| \geq \frac{1}{q_1q_2} - \frac{c_1}{q_1^2} \\ &= \frac{1}{q_1q_2} - \frac{1}{3q_1^2} \geq \frac{1}{q_1q_2} - \frac{2}{3q_1q_2} = \frac{1}{3q_1q_2} \geq \frac{1}{3q_2^2}.\end{aligned}$$

On the other hand:

$$\left|\alpha - \frac{p_2}{q_2}\right| \leq \left|\frac{p_1}{q_1} - \frac{p_2}{q_2}\right| + \left|\alpha - \frac{p_1}{q_1}\right| \leq \frac{1}{q_1q_2} + \frac{1}{3q_1^2} \leq \frac{10}{3q_2^2}.$$

This proves the lemma. ∎

Proof of Theorem 2.3: Using Lemma 4.3, take a (large) q such that $\eta = \alpha - p/q = \theta/q^2$ for an appropriate p with $(p,q) = 1$, where $1/4 \leq |\theta| \leq 3$. Define f by $f(x) = P(x)f_1(xf_2(x\alpha))$. Put $\beta = f_2(x_0)$.

Assume first that β is rational, say $\beta = a/b$. To prove the theorem, it suffices to show that every interval $[u,v]$ contains at least one point of the form $\{f(n)\}$ for some $n = b(mq+k)$, where $2M \geq m \geq M = q^{(l-1)/(l+1)}/\log q$ and $|k| \leq q/2$, $kp \equiv k_0 = \left[\frac{qx_0}{b}\right] \pmod{q}$. Denoting by N the number of such n's, we employ Lemma 4.2 with $\delta = \frac{\Delta}{5}$ and $\Delta = v - u$ to get

$$N \geq \frac{4M\Delta}{5} - 2\sum_{j=1}^{\infty} \min\left\{\Delta,\ \frac{25}{\Delta j^2},\ \frac{25}{\Delta^2 j^3}\right\} \cdot \left|\sum_{M \leq m < 2M} e(jf(n))\right|.$$

For $j > 8/\Delta^2$ we estimate the sum trivially and obtain

$$N \geq \frac{M\Delta}{2} - 2\sum_{j=1}^{8/\Delta^2}\left|\sum_{M\leq m<2M} e(jf(n))\right| \cdot \min\left\{\Delta, \frac{25}{\Delta j^2}\right\}.$$

We now want to show that $\sum_{M\leq m<2M} e(jf(n)) = o(M)$, which will prove that $N \gg M$, and thus establish the theorem in this case. In fact

$$\begin{aligned} f(n) &= P(n)f_1\left(nf_2\left(\frac{bpk}{q} + n\eta\right)\right) \\ &= P(n)f_1\left(nf_2\left(\frac{bk_0}{q} + n\eta\right) - n\beta\right) \\ &= P(n)f_1(ng(n)), \end{aligned}$$

where

$$\begin{aligned} g(n) &= f_2\left(\frac{bk_0}{q} + n\eta\right) - \beta \\ &= \frac{1}{l!}f_2^{(l)}(x_0)\left(n\eta + \frac{bk_0}{q} - x_0\right)^l (1+o(1)) \sim \left(\frac{M}{q}\right)^l. \end{aligned} \tag{4.5}$$

Hence

$$ng(n) \sim \frac{M^{l+1}}{q^{l-1}} \sim \frac{1}{(\log q)^{l+1}}.$$

Let s be the smallest integer $\geq \frac{d}{l-1} + \frac{1}{2(l+1)}$ such that $a_s = \frac{1}{s!}f_1^{(s)}(0) \neq 0$, and let r be the smallest integer satisfying

$$r \geq \frac{1}{2} + \frac{2dl}{l-1}.$$

Then we have:

$$r \leq d + s(l+1).$$

Now

$$\frac{d^r(f(n))}{dm^r} = a_s \frac{d^r}{dm^r}\left(P(n)\cdot\left[\frac{nf_2^{(l)}(0)}{l!}(n\eta + bk_0/q - x_0)^l\right]^s\right)\cdot(1+o(1)),$$

so that

$$\frac{d^r(f(n))}{dm^r} \sim \frac{(Mq)^d}{M^r}(\log q)^{-(l+1)s} \sim M^{d-r+\frac{d(l+1)}{l-1}}(\log M)^{\left(\frac{d}{l-1}-s\right)(l+1)}. \tag{4.6}$$

Denoting the right hand side of (4.6) by λ, we see that

$$M^{-3/2-\epsilon} \ll \lambda \leq M^{-1/2}.$$

Note also that $r \geq 2d+1 \geq 3$. Employing Lemma 4.1 (with r in place of j), we obtain

$$\left| \sum_{M \leq m < 2M} e(jf(n)) \right| \ll M^{1-1/(2J-4)} = o(M).$$

Now we shall deal with the case of irrational β. Write

$$\beta q = p_1/q_1 + \eta_1,$$

where $|\eta_1| \leq 1/q_1 Q$, $q_1 \leq Q = q^{(l-1)/(2l+2)}$, $(p_1, q_1) = 1$. Let $|k| \leq q/2$, $kp \equiv k_0 \pmod{q}$, $k_0 = [qx_0]$, and m_0 is an integer satisfying

$$\|m_0 q_1 \eta_1 + (qq_1 m_0 + k) g(qq_1 m_0 + k) + k\beta - x_1\| \ll q_1 \eta_1 + q_1 (m_0 q_1)^l q^{1-l},$$

where $g(n)$ is as in (4.5) with $b = 1$, x_1 and m_0 to be specified subsequently. The existence of such an m_0 of order of magnitude as in the following is ensured by the equality

$$\frac{d}{dm}\left[m q_1 \eta_1 + (qq_1 m + k) \cdot g(qq_1 m + k)\right] = q_1 \eta_1 + O\left(q_1 (m_0 q_1)^l q^{1-l}\right).$$

Take $n = qq_1(m_0 + m) + k$.

Suppose first that $|\eta_1| \leq \frac{\log q}{Q^2}$. Then we select $m_0 \sim \frac{Q^2 \log q}{q_1}$ and $m \sim M = m_0(\log q)^{-l-2}$. Let r be the minimal integer for which $(qq_1)^d m_0^{d-r+\frac{1}{2}} \leq 1$. Choose a point x_1 such that $f_1^{(r)}(x_1) \gg 1$. Put

$$g_1(m) = ng(n) - (m_0 qq_1 + k) \cdot g(m_0 qq_1 + k).$$

Since

$$f(n) = P(n) \sum_i \frac{f_1^{(i)}(x_1)}{i!} [m q_1 \eta_1 + g_1(m)]^i + R(m)$$

for some small error term $R(m)$, we have

$$\begin{aligned} \lambda = \frac{d^r(f(n))}{dm^r} &= f_1^{(r)}(x_1) P(n) [q_1 \eta_1 + \frac{dg_1(m)}{dm}]^r \cdot [1 + O(1/\log q)] \\ &\sim (m_0 qq_1)^d m_0^{-r} (\log q)^{(l+1)r}, \end{aligned}$$

whence $M^{-3/2-\epsilon} \ll \lambda \leq M^{-1/2}$. Similarly to the case of rational β, we arrive at the inequality $N \gg M$.

It remains to treat the case $|\eta_1| > \frac{\log q}{Q^2}$. Take $m_0 \sim \frac{\log q}{q_1\eta_1}$ and $m \sim M = \frac{m_0}{\log^2 q}$. Let r be the smallest integer such that $(qq_1)^d M^{d-r+\frac{1}{2}} \leq 1$, and choose x_1 as in the preceding case. The conclusion of the proof is as before. ∎

Proof of Theorem 2.4: Denote:

$$K = \left\{k\colon |k| \leq \frac{q}{2},\ q^{-2\epsilon_0} \leq \left\|\frac{kp}{q} - x_0\right\| \leq 2q^{-2\epsilon_0}\right\}.$$

For $k \in K$ write $qf_2\left(\frac{kp}{q}\right) = \frac{p_1}{q_1} + \eta_1$ with $(p_1, q_1) = 1$, $1 \leq q_1 \leq Q = q^{\epsilon_0/2}$ and $|\eta_1| \leq \frac{1}{q_1 Q}$. Denote also

$$A = A(k) = \left\{n = mqq_1 + k\colon m \sim M = Q^{18/19}\right\}.$$

Obviously, $|A| \sim q^{9\epsilon_0/19}$. Putting $g(n) = f_2(n\alpha) - f_2\left(\frac{kp}{q}\right)$, and using the equality

$$f_2'\left(\frac{kp}{q}\right) = \frac{1}{(l-1)!} f_2^{(l)}(x_0)\left(\frac{k_1}{q} - x_0\right)^{l-1}(1 + o(1)),$$

where $k_1 \equiv kp \pmod q$ and $\left|\frac{k_1}{q} - x_0\right| \sim q^{-2\epsilon_0}$, we get

$$g(n) = n\eta f_2'\left(\frac{kp}{q}\right) + O(n^2\eta^2) \sim q^{-2\epsilon_0(l-1)} n\eta.$$

Therefore

$$ng(n) \sim |M^2 q^2 q_1^2 q^{-2\epsilon_0(l-1)}\eta| \leq M^2 Q^2 q^{-2\epsilon_0} \leq Q^{-2/19}.$$

Also, $|mq_1\eta_1| \leq M/Q \sim Q^{-1/19}$. For any $n \in A$ we have:

$$\begin{aligned} f_1(nf_2(n\alpha)) &= f_1\left(nf_2\left(\frac{kp}{q} + n\eta\right)\right) = f_1\left(nf_2\left(\frac{kp}{q}\right) + ng(n)\right) \\ &= f_1\left(kf_2\left(\frac{kp}{q}\right) + mq_1\left(\frac{p_1}{q_1} + \eta_1\right) + ng(n)\right) \\ &= f_1\left(kf_2\left(\frac{kp}{q}\right) + mq_1\eta_1 + ng(n)\right). \end{aligned}$$

Put:

$$K_1 = \{k \in K\colon |q_1\eta_1| \geq 1/(q_1 Q^{1/20})\}.$$

We want to show that $|K_1| \gg q^{1-2\epsilon_0}$, namely $|K_1| \sim |K|$. To this end, it clearly suffices to prove that $|K - K_1| = o(q^{1-2\epsilon_0})$. In fact, we need to find an upper bound for the number of those $k \in K$ satisfying $\left\|qq_1 f_2\left(\frac{k_1}{q}\right)\right\| \le \frac{1}{q_1 Q^{1/20}}$ for some $q_1 \le Q$ (where $k_1 \equiv kp (\mathrm{mod}\, q)$) and $\frac{k_1}{q} \in [x_0 - 2q^{-2\epsilon_0}, x_0 - q^{-2\epsilon_0}]$. Employing Lemma 4.2 with $\Delta = 4\delta = 1/(q_1 Q^{1/20})$, we obtain:

$$|K - K_1| \ll \sum_{q_1 \le Q} \sum_{k \in K} \Delta + \sum_{q_1 \le Q} \sum_{j=1}^{\infty} \min\left\{\Delta, \frac{1}{\Delta j^2}\right\} \left| \sum_{k \in K} e\left(jqq_1 f_2\left(\frac{k_1}{q}\right)\right) \right|$$

$$= o(|K|) + \sum_{q_1 \le Q} \sum_{j=1}^{\infty} \min\left\{\Delta, \frac{1}{\Delta j^2}\right\} \left| \sum_{qx_0 - 2q^{1-2\epsilon_0} \le k \le qx_0 - q^{1-2\epsilon_0}} e\left(jqq_1 f_2\left(\frac{k}{q}\right)\right) \right|.$$

To estimate the inner sum over k use Lemma 4.1 with $j = 2$ and

$$\lambda \sim \left| \frac{d^2}{dk^2}\left(jqq_1 f_2\left(\frac{k}{q}\right)\right) \right| = \frac{jq_1}{q} \left| f_2''\left(\frac{k}{q}\right) \right| \sim jq_1 q^{-1-2(l-2)\epsilon_0}.$$

Therefore:

$$\left| \sum_{qx_0 - 2q^{1-2\epsilon_0} \le k \le qx_0 - q^{1-2\epsilon_0}} e\left(jqq_1 f_2\left(\frac{k}{q}\right)\right) \right| \ll q^{1-2\epsilon_0} \sqrt{jq_1 q^{-1-2(l-2)\epsilon_0}} + \frac{1}{\sqrt{jq_1 q^{-1-2(l-2)\epsilon_0}}},$$

so that

$$\begin{aligned} |K - K_1| &= o(|K|) + O\left(q^{1-2\epsilon_0} Q \sqrt{Q^{41/20} q^{-1-2(l-2)\epsilon_0}} + \sqrt{q^{1+2(l-2)\epsilon_0} Q^{-1/20}}\right) \\ &= o(|K|) + o(q^{1-2\epsilon_0}) = o(|K|) \end{aligned}$$

(here we have invoked the condition on ϵ_0).

Denote $B = \bigcup_{k \in K_1} A(k)$ and $B_I = \{n \in B\colon \{f(n)\} \in I\}$. The foregoing estimates yield $|B| \gg q^{1-2\epsilon_0} q^{9\epsilon_0/19} = q^{1-29\epsilon_0/19}$. We ought to show that the same holds with B replaced by B_I, namely $|B_I| \gg q^{1-29\epsilon_0/19}$.

Let $j_0 = j_0(k)$ be the smallest integer satisfying $(qq_1)^d |q_1 \eta_1|^{j_0 - d} \le M^{-1/3}$. Since $(qq_1)^d |q_1 \eta_1|^{j_0 - d} \le (qQ)^d Q^{d - j_0}$, one verifies easily that $j_0 \le [2d(1 + 1/\epsilon_0) + 4/3]$. On the other hand, since

$$M^{-1/3} \ge (qq_1)^d |q_1 \eta_1|^{j_0 - d} \ge (qq_1)^d (q_1 Q^{1/20})^{d - j_0},$$

we have $j_0 > 9d \geq 9$. Set:

$$j_1 = j_1(k) = \min\left\{j: j \geq j_0 - d, \left|f_1^{(j)}\left(kf_2\left(\frac{kp}{q}\right)\right)\right| \geq \frac{a_0}{3}\right\}.$$

The conditions of the theorem imply that $j_0 - d \leq j_1 \leq j_0 - d + 2$. Taking $r = j_1 + d$, we get

$$\begin{aligned}\lambda &\sim \left|\frac{d^r f(n)}{dm^r}\right| = \left|\sum_{i=0}^{d}\binom{r}{i}\frac{d^i P(n)}{dm^i}\frac{d^{r-i}}{dm^{r-i}}\left(f_1\left(kf_2\left(\frac{kp}{q}\right) + mq_1\eta_1 + ng(n)\right)\right)\right| \\ &\sim \left|\sum_{i=0}^{d}\binom{r}{i}\frac{d!}{(d-i)!}(qq_1)^i n^{d-i}|q_1\eta_1|^{r-i} f_1^{(r-i)}\left(kf_2\left(\frac{kp}{q}\right)\right)(1+o(1))\right| \\ &\sim (qq_1)^d |q_1\eta_1|^{r-d}\left|f_1^{(j_1)}\left(kf_2\left(\frac{kp}{q}\right)\right)\right| \\ &\sim (qq_1)^d |q_1\eta_1|^{j_1} \leq M^{-1/3}\end{aligned}$$

and

$$\lambda \geq (qq_1)^d |q_1\eta_1|^{j_0-d+2} \geq M^{-1/3}|q_1\eta_1|^3 \geq M^{-1/3}Q^{-3\cdot 21/20} \geq M^{-11/3}.$$

By Lemma 4.2, with $\Delta = 5\delta = |I|$,

$$|B_I| \geq \delta \sum_{k\in K_1}\sum_{m\sim M} 1 - 100\sum_{j=1}^{\infty}\min\left\{\Delta, \frac{1}{\Delta j^2}\right\}\cdot\sum_{k\in K_1}\left|\sum_{m\sim M} e(jf(n))\right|.$$

Employing Lemma 4.1 (with our r in place of j from the lemma), we finally obtain

$$\begin{aligned}|B_I| &\gg Mq^{1-2\epsilon_0} \\ &+ O\left(\sum_{j=1}^{\infty}\min\left\{\Delta, \frac{1}{\Delta j^2}\right\} q^{1-2\epsilon_0} M[(M^{-\frac{1}{3}}j)^{\frac{1}{J-2}} + M^{-\frac{2}{J}} + (M^{-\frac{11}{3}+4-\frac{8}{J}})^{-\frac{2}{J}}]\right) \\ &\gg Mq^{1-2\epsilon_0} = q^{1-\frac{29\epsilon_0}{19}},\end{aligned}$$

which completes the proof. ∎

Proof of Corollary 2.2: Follows from Theorem 2.4 (and Lemma 4.3). ∎

Proof of Corollary 2.3: Write $\alpha = \frac{p}{q} + \eta$ with $(p,q) = 1$, $N^{u(1-\epsilon_0)} \leq q \leq N^{1-\epsilon_0}$ and $|\eta| \leq \frac{1}{qN^{1-\epsilon_0}} \leq \frac{1}{q^2}$. In view of Theorem 2.4

$$|\{n \leq N: \langle f(n)\rangle \in I\}| \geq |\{n \leq q^{1+\epsilon_0}: \langle f(n)\rangle \in I\}| \gg q^{1-2\epsilon_0} \geq N^{u(1-2\epsilon_0)}.$$ ∎

Proof of Proposition 2.3: 1. We proceed as in the proof of Theorem 2.4, but make our choices with more care. Taking $\varepsilon_0 = \frac{2}{9}$, $Q = \varepsilon_1 q^{1/9}$ and $M = \varepsilon_1 Q$ with sufficiently small ε_1 yields the desired conclusion.

2. Split the sequence into the $2q$ subsequences obtained by restricting n to the various congruence classes modulo $2q$. Letting $n = 2qm + k$ for some fixed $0 \leq k < q$, we have

$$n \cos(n \cos n\alpha) = (2qm + k) \cos((2qm + k)\gamma),$$

where $\gamma = \cos \frac{kp}{q}\pi$. Clearly, γ is an algebraic number, so it is not a rational multiple of π unless $\gamma = 0$, namely $kp \equiv -\frac{q}{2} (\mathrm{mod}\, q)$. Now if $\gamma = 0$ then the subsequence in question consists of integers, whereas otherwise it easily follows from [8, Th. 4.1] to be u.d. Now if q is odd then $\gamma \neq 0$ for each k, so that our sequence is a disjoint union of $2q$ subsequences, each of which is u.d., whence the sequence itself is u.d. as well. If q is even then some of the $2q$ subsequences are u.d. while the others are identically 0 modulo 1. Thus our sequence is distributed modulo 1 according to the corresponding convex combination of the Lebesgue measure on $[0, 1)$ and the Dirac measure at 0. ∎

Proof of Proposition 2.4: Take a sequence n_k with $\|n_k \frac{\alpha}{2\pi}\| < \frac{1}{n_k}$ for each k. For an arbitrary fixed positive integer h we have

$$\begin{aligned} \log hn_k \cdot \cos hn_k\alpha &= (\log h + \log n_k) \left(1 + O\left(\frac{1}{n_k^2}\right)\right) \\ &= \log n_k + \log h + O\left(\frac{\log n_k}{n_k^2}\right). \end{aligned}$$

Passing to a subsequence we may assume that the sequence $\log n_k$ converges modulo 1, so that

$$\log hn_k \cdot \cos hn_k\alpha \underset{k\to\infty}{\longrightarrow} \log h + b$$

for a certain b. As the sequence $(\log h)_{h=1}^{\infty}$ is clearly dense modulo 1, this proves the proposition. ∎

5. Proofs of the negative results

Proof of Theorem 2.5: Since $\lim_{n\to\infty} \mu_n = +\infty$, one can choose a sequence

$$n_0 = 0 < n_1 < n_2 < n_3 < \cdots$$

such that $|\mu_n| > k^2$ for all $n > n_k$. For all $k \geq 0$, put $M_k = \mathbb{N} \cap [n_k + 1, n_{k+1}]$. (Thus $\{M_k\}_{k\geq 1}$ is a partition of $\mathbb{N}$ into finite sets of consecutive integers.)

Denote by $C_{\mathrm{p}}(\mathbb{R})$ the set of continuous 1-periodic functions on $\mathbb{R}$. Let $f_0(x) \in C_{\mathrm{p}}(\mathbb{R})$ be an arbitrary function such that (2.4) holds for all $n \in M_0$. Proceeding by induction on k, one constructs the functions $f_k(x) \in C_{\mathrm{p}}(\mathbb{R})$ as follows. First define f_k on a finite set

$$f_k(\alpha_n) = \begin{cases} 0, & n \leq n_k; \\ \mu_n^{-1}\langle \beta_n - \sum_{j=0}^{k-1} \mu_n f_j(\alpha_n)\rangle, & n \in M_k; \end{cases}$$

(where $\langle x\rangle = x - [x]$ stands for the fractional part of x), and then extend f_k to be piecewise linear and 1-periodic on $\mathbb{R}$. Thus we have $0 \leq f_k(x) < \frac{1}{k^2}$ (since $|\mu_n| > k^2$ for all $n > n_k$).

The function $f(x) = \sum_{k=0}^{\infty} f_k(x)$ is easily seen to satisfy the conditions of the theorem. ∎

Proof of Theorem 2.6: Without loss of generality we assume that $d < \frac{1}{2}$. Fix a $\phi(x) \in C^\infty(\mathbb{R})$ supported by the interval $\left[-\frac{1}{2}, +\frac{1}{2}\right]$ such that $0 \leq \phi(x) \leq 1$ and $\phi(0) = 1$. For any s, $0 < s < 1$, put

$$\phi_s(x) = \phi(s, x) = \sum_{i=-\infty}^{\infty} \phi\left(\frac{x+i}{s}\right) \in C^\infty(\mathbb{R}).$$

It is clear that $\phi_s(x)$ is 1-periodic and that, for every $k \geq 0$,

$$\max_x |\, \phi_s^{(k)}(x)\, | = s^{-k} \max_x |\, \phi^{(k)}(x)\, |$$

where $\phi^{(k)}(x) = \left(\frac{d}{dx}\right)^k \phi(x)$.

The required function f is defined by the series

$$f(x) = \sum_{i=1}^{\infty} U_i(x),$$

where

$$U_i(x) = b_i \phi(s_i, x - \alpha_i),$$

$$s_i = \frac{d}{2} \cdot (2i)^{-t},$$

and the constants b_i are defined inductively:

$$b_1 = \frac{\langle \beta_1 \rangle}{\mu_1},$$

and

$$b_{n+1} = \frac{\langle \beta_n - \sum_{i=1}^{n} b_i \phi(s_i, \alpha_{n+1} - \alpha_i) \rangle}{\mu_{n+1}} = \frac{\langle \beta_n - \sum_{i=1}^{n} U_i(\alpha_{n+1}) \rangle}{\mu_{n+1}}, \qquad n \geq 1.$$

Denote

$$g_m(x) = \sum_{i=2^{m-1}}^{2^m - 1} b_i \phi(s_i, x - \alpha_i) = \sum_{i=2^{m-1}}^{2^m - 1} U_i(x), \qquad m \geq 1.$$

Clearly $f(x) = \sum_{m \geq 1} g_m(x)$. Moreover, the addends in the sum for $g_m(x)$ have pairwise disjoint supports (due to condition 2 and our choice of s_i) and therefore for each $r = 0, 1, \ldots, k$ and $m \geq 1$, we have

$$\begin{aligned}\max_x |\, g_m^{(r)}(x) \,| &\leq \max\{b_i |\, 2^{m-1} \leq i \leq 2^m - 1\} \cdot (s_{2^m})^{-r} \max_x |\, \phi^{(r)}(x) \,| \\ &\leq c_1 \cdot \max\{\mu_i^{-1} |\, 2^{m-1} \leq i \leq 2^m - 1\} \cdot 2^{mrt} \\ &\leq c_2 2^{mrt - mp} \leq c_2 2^{-m(p-kt)},\end{aligned}$$

where the constants c_1, c_2 do not depend on m or x.

Thus $f(x) = \sum_{m \geq 1} g_m(x)$ converges in the C^k-norm (by condition 4) and therefore $f(x)$ is in C^k. It is also clear that $f(x)$ is 1-periodic because each $U_i(x)$ and $g_m(x)$ is. Finally, the choice of the b_i assures that (2.4) holds. ∎

Proof of Corollary 2.5: Choose any p such that

$$\liminf_{n \longrightarrow \infty} \frac{\log |\mu_n|}{\log n} > p > k,$$

then choose t such that

$$1 < t < \frac{p}{k}.$$

Now we can apply Theorem 2.6. ∎

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